Algebraic Geometry Example Sheet 3: Lent 2026

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `hk439@cam.ac.uk`. In all questions, k is an algebraically closed field of characteristic 0.

1. Let U be the complement of the origin in $\mathbb{A}^2$. Consider a rational map

$$\varphi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$$

that is regular on all points of U. Prove that φ is a morphism. Comment on whether U can be an affine variety.

2. It is true that every pair of (possibly singular) curves in $\mathbb{P}^2$ have non-empty intersection. Use this to show that any morphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ is constant[1]. Deduce that $\mathbb{P}^2$ is not isomorphic to a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

3. Determine the singular points of the surface $Z(xy^2 - z^3) \subset \mathbb{P}^3$, where $\mathbb{P}^3$ has homogeneous coordinates x, y, z, w. Find the dimension of the tangent space at all singularities.

4. Consider the curve $C := Z(xy^2 - z^3) \subset \mathbb{P}^2$. Show that $(t : u) \mapsto (t^3 : u^3 : tu^2)$ defines a morphism $\varphi : \mathbb{P}^1 \to C$. Write down a rational map $\psi : C \dashrightarrow \mathbb{P}^1$, a morphism on $U = C \setminus \{(1 : 0 : 0)\}$ which is inverse to φ on U. What is the geometric meaning of ψ? Show that ψ does not extend to a morphism.

5. Let $A = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$. A polynomial $f \in A$ is *bihomogeneous* of bidegree (d, e) if it is homogeneous of degree d in the x_i, treating the y_i as scalars, and homogeneous of degree e in the y_i, treating the x_i as scalars. Show that if f is bihomogeneous then the vanishing locus $Z(f)$ is a well-defined subset of $\mathbb{P}^m \times \mathbb{P}^n$.

6. Show that the closed sets of the Zariski topology on $\mathbb{P}^m \times \mathbb{P}^n$ via the Segre embedding are precisely the vanishing loci of bihomogeneous polynomials.

7. Define plane curves $C_1 := Z(x^8 + y^8 + z^8)$ and $C_2 := Z(x^4 + y^4 + z^4)$, and let $\varphi : C_1 \to C_2$ be the morphism $(x : y : z) \mapsto (x^2 : y^2 : z^2)$. Determine the degree of this morphism and compute the ramification indices for all points of C_1.

8. Define the plane cubic $C := Z(zy^2 - x^3 + 3xz^2)$. Show that C is irreducible and smooth. Let $\pi_x : C \to \mathbb{P}^1$ be the restriction of the projection $(x : y : z) \mapsto (x : z)$, and $\pi_y : C \to \mathbb{P}^1$ the restriction of the projection $(x : y : z) \mapsto (y : z)$. Determine the degree and all ramification degrees of the morphisms π_x and π_y.

9. Let $C \subset \mathbb{P}^2$ be a (smooth irreducible) degree d curve. Let P be a point on C and define $\pi_P : C \to \mathbb{P}^1$ to be the projection from P restricted to C. Prove that for all but finitely many points Q on $\mathbb{P}^1$, the set $\pi_P^{-1}(Q)$ contains $d - 1$ points. Deduce the degree of π.

[1] In fact such a morphism would have smooth fibers $\varphi^{-1}(P)$ for all but finitely many $P \in \mathbb{P}^1$: this is a so-called *Bertini's theorem*, which is proved by studying the action of $d\varphi_P$ on the three relevant tangent spaces. So we don't really need to allow singular curves in this hypothesis.